On the dynamics of the Heckscher-Ohlin theory

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5 of June 2009

Literature

- Oniki - Uzawa (1965)
- Stiglitz (1970)
- Chen (1992)
- Baxter (1992)
- Jensen and Wang (1997)
- Ventura J. (1997)
- Atkenson Kehoe (2000)
- Bajona and Kehoe (2006)
- Gaitan Roe (2006)

Environment

- 2x2x2 Heckscher-Ohlin model
- 2 Countries $i = \{N, S\}$ 2 Tradable goods y_j^i $j = \{1, 2\}$ 2 Immobile Factors of production K_j^i and L_j^i.
- 1 Final good Y^i - Not tradable
- x_j^i the demand of intermediate inputs 1 and 2.
- Identical Cobb Douglas Technologies
- c^i and i^i the consumption and investment of the household.
- Prices: p_j the price of the intermediate inputs and r^i and w^i the rental price of capital and the wage rate. We normalize the price of the final good to 1.
- Trade is balanced date by date $\sum_j p_j x_j^i = \sum_j p_j y_j^i$
- All variables per capita, intensive form.
- Assume that $k^N(0) > k^S(0)$

Household-firm in each country
The problem they solve is the following:

$$\max \int_0^\infty \exp\left(-\rho t\right) \log\left(c^i\left(t\right)\right) dt$$

subject to:

$$c^i\left(t\right) + i^i\left(t\right) = G\left(k^i; p\right)$$
$$\dot{k}^i\left(t\right) = i^i\left(t\right) - \delta k^i\left(t\right)$$
$$k^i\left(0\right) \text{ given}$$

- $G\left(k^i;p\right)$ solves the following static sub problem:

$$G\left(k^i;p\right) = \max\left(x_1^i\right)^\gamma \left(x_2^i\right)^{1-\gamma} \tag{1}$$

subject to:

$$\sum_j p_j x_j^i = \sum_j p_j y_j^i \tag{2}$$

$$y_j^i = \left(k_j^i\right)^{\theta_j} \left(l_j^i\right)^{1-\theta_j} \qquad j = \{1,2\} \tag{3}$$

$$k^i = \sum_j k_j^i \tag{4}$$

$$1 = \sum_j l_j^i \tag{5}$$

$$k_j^i \geq 0,\ l_j^i \geq 0,\ y_j^i \geq 0 \tag{6}$$

Note that we are allowing for corner solutions in which one or both of the countries specialize.

The price of the intermediate goods $p = \{p_1, p_2\}$ are determined in the world commodity market and are such that the world commodity market clear:

$$\sum_i l^i x_j^i \left(k^i; p\right) = \sum_i l^i y_j^i \left(k^i; p\right) \qquad j = \{1, 2\} \tag{7}$$

where $x_j^i \left(k^i; p\right)$ and $y_j^i \left(k^i; p\right)$ are the optimal conditional demand and supply of intermediate inputs in each country.

How to solve this problem.
Strategy:

- Dynamic problem
- Characterize set of endowment distributions consistent with FPE.
- Divide the plane into different regions according to initial endowment distributions.
- Characterize the dynamics in each region.

- Dynamic problem

Current value Hamiltonian in each country.

$$\tilde{H}^i = \log\left(c^i(t)\right) + q^i(t)\left[G\left(k^i(t); p(t)\right) - c^i(t) - \delta k^i(t)\right]$$

$q^i(t)$ is the current value co-state variable.
The necessary first order conditions are:

$$\left(c^i\right)^{-1} = q^i \quad (8)$$

$$\dot{q}^i = -\left(G_{k^i}\left(k^i; p\right) - (\rho + \delta)\right) q^i \quad (9)$$

$$\dot{k}^i = G\left(k^i; p\right) - c^i - \delta k^i \quad (10)$$

$$\lim_{t\to\infty} e^{-\rho t} k^i q^i = 0 \quad (11)$$

G_{k^i} is the partial derivative of total production with respect to k^i, the marginal product of capital in country i

The following differential equations and the commodity market equilibrium condition (7) describe the equilibrium dynamics of this model:

$$\begin{aligned}
\dot{c}^N &= c^N \left(G_{k^N}\left(k^N; p\right) - (\rho + \delta) \right) \\
\dot{c}^S &= c^S \left(G_{k^S}\left(k^S; p\right) - (\rho + \delta) \right) \\
\dot{k}^N &= G\left(k^N; p\right) - c^N - \delta k^N \\
\dot{k}^S &= G\left(k^S; p\right) - c^S - \delta k^S
\end{aligned}$$

Characterize set of endowment distributions consistent with FPE.
Assume that FPE holds and solve for the distribution of endowments.
FPE set:

$$V\left(k^N, k^S\right) = \left\{\left(k^N, k^S\right) \text{ s.t. } \tilde{k}_2 \leq k^N \leq \tilde{k}_1 \text{ and } \tilde{k}_2 \leq k^S \leq \tilde{k}_1\right\}$$

where $k^i = K^i / L^i$, $\tilde{k}_2$ and $\tilde{k}_1$ are capital labor ratios that characterize the lower and upper bound of the cone of diversification.
The solutions for $\tilde{k}_j$ are:

$$\tilde{k}_j = \left(\frac{\theta_j}{1-\theta_j}\right)\left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right)\frac{k}{2}$$

where $\tilde{\gamma} \equiv \gamma\theta_1 + (1-\gamma)\,\theta_2$, and $k = \sum_i k^i$.

Divide the plane into different regions according to the initial endowment distributions.

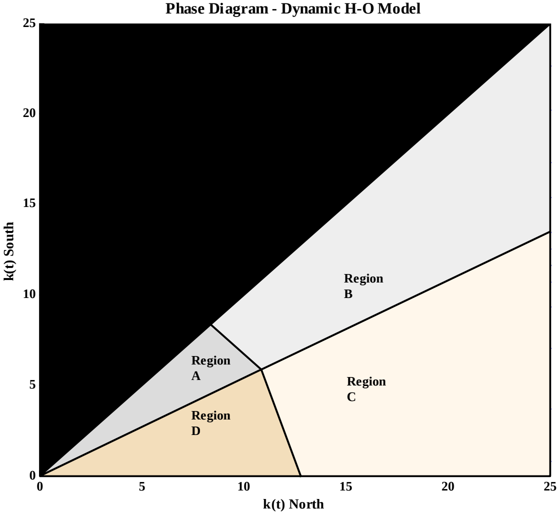


Figure 1

Characterize the dynamics in each region.

- $k^{i}(0) \in V\left(k^{N}, k^{S}\right)$, Inside the Cone of Diversification. Regions A and B
- $k^{S}(0) \notin V\left(k^{N}, k^{S}\right)$, Outside the Cone of Diversification. Regions C and D

Inside the Cone of Diversification.

Initial distribution of endowments

$$k^N(0) - k^S(0) \leq \left(\frac{\theta_1}{1-\theta_1} - \frac{\theta_2}{1-\theta_2} \right) \frac{1-\tilde{\gamma}}{\tilde{\gamma}} \frac{k(0)}{2} \tag{12}$$

From the necessary first order conditions, and assuming that FPE holds, the growth rate of consumption is the same in both countries. This implies that the relative consumption of the households in each country stays constant over time.

$$\frac{c^S(0)}{c^N(0)} = \varpi \tag{13}$$

$$\frac{c^S(t)}{c^N(t)} = \frac{c^S(0)\, e^{(R(t)-(\rho+\delta))t}}{c^N(0)\, e^{(R(t)-(\rho+\delta))t}} = \varpi$$

Dynamics

Steady state.

Stability: Linearize the system in the neighborhood of the steady state.

From the resulting characteristic equation solve for the 3 eigenvalues.

Steady state

$$r^* = \rho + \delta$$

The steady state world aggregate stock of capital and aggregate consumption are given by:

$$\frac{k^*}{2} = \frac{\tilde{\gamma}}{1-\tilde{\gamma}}\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \quad (14)$$

$$\frac{c^*}{2} = \left(\frac{\rho+\delta\left(1-\tilde{\gamma}\right)}{1-\tilde{\gamma}}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \quad (15)$$

where c is the world per capita consumption: $c^* = c^{N*} + c^{S*}$ and s^* is the steady state savings rate.

Proposition 1 *There exists a unique steady state level for the world aggregate variables. The steady state level for the country aggregates is a function of the initial wealth distribution, hence there are and infinite number of such steady states.*

$$c^{N*} = \frac{1}{1+\varpi} c^* \tag{16}$$

$$c^{S*} = \frac{\varpi}{1+\varpi} c^* \tag{17}$$

$$k^{N*} = \frac{1}{\rho}\left(\frac{1}{1+\varpi} c^* - w^*\right) \tag{18}$$

$$k^{S*} = \frac{1}{\rho}\left(\frac{\varpi}{1+\varpi} c^* - w^*\right) \tag{19}$$

Note that the steady state levels for the country aggregates are a function of the initial wealth distribution, hence they are not unique.

Do we stay inside the cone? Let $z^S(t) = k^S(t)/k(t)$
Suppose that at $t = 0$ we start in the boundary of the cone, then

$$z^S(0) = \left(\frac{\theta_2}{1-\theta_2}\right)\left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right)\frac{1}{2}$$

If $\dot{z}^S(t) > 0$ countries move strictly inside the cone.

Lemma (1)

Suppose that countries factor supplies belong to the lower bound of the cone of diversification. If $k(0) < k^*$ then $\dot{z}^S(t) > 0$ and if $k(0) > k^*$ then $\dot{z}^S(t) < 0$.

This implies that if we are in region A then the economies stay in region A, however if we are in region B, the economies might leave the cone of diversification. In particular, if they start in the boundary of the cone, the ray dividing region B from region C, they move to region C.

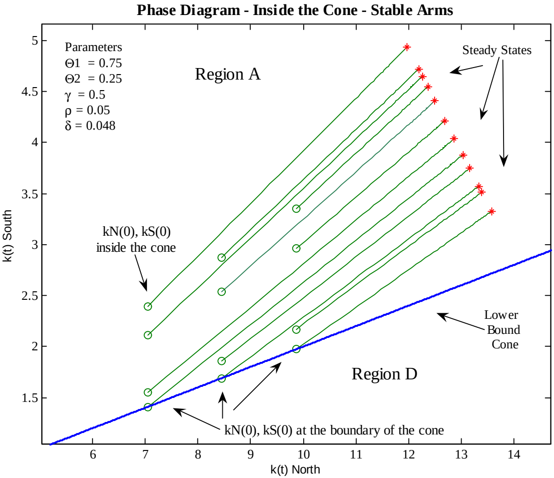

Phase Diagram - Inside the Cone - Stable Arms
Parameters
Θ1 = 0.75
Θ2 = 0.25
γ = 0.5
ρ = 0.05
δ = 0.048
Region A
Steady States
kN(0), kS(0)
inside the cone
Lower
Bound
Cone
Region D
kN(0), kS(0) at the boundary of the cone
5
4.5
4
3.5
3
2.5
2
1.5
6
7
8
9
10
11
12
13
14
k(t) South
k(t) North

Figure 2

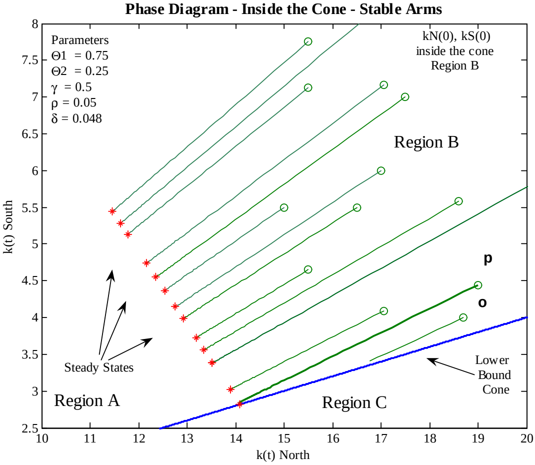
Slope of the saddle path.
Compare this slope with the slope of the cone of diversification.

Lemma (2)

The slope of the saddle path is larger than the slope of the lower boundary of the cone.

$$\lim_{t\to\infty}\frac{\dot{k}^S(t)}{\dot{k}^N(t)} = \psi'_{k^S}\left(k^{N*}\right) = \frac{-\alpha(1-\varpi)-\varpi\tilde{\lambda}}{\alpha(1-\varpi)-\tilde{\lambda}}$$

$$\psi'_{k^S}\left(k^{N*}\right) > \frac{(1-\tilde{\gamma})\,\theta_2}{(1-\theta_2)\,\tilde{\gamma}2-(1-\tilde{\gamma})\,\theta_2}$$



Phase Diagram - Inside the Cone - Stable Arms
Parameters
Θ1 = 0.75
Θ2 = 0.25
γ = 0.5
ρ = 0.05
δ = 0.048
kN(0), kS(0)
inside the cone
Region B
Region B
p
o
Steady States
Region A
Region C
Lower
Bound
Cone
k(t) South
k(t) North

Figure 3

Outside the Cone of Diversification.

Initial distribution of endowments

$$\frac{k^S(0)}{k^N(0)} < \frac{\theta_2(1-\tilde{\gamma})}{\theta_2(1-\tilde{\gamma}) + 2(\tilde{\gamma}-\theta_2)} \quad (20)$$

South will only produce and export the labor intensive intermediate good 2 and import the capital intensive good. The problem of the agent in south has three binding constraints which imply:

$$k_1^S = 0, l_1^S = 0, y_1^S = 0$$

Dynamics

The laws of motion governing transition for both countries are:

$$\dot{c}^N = \left(G_{K^N}\left(k^N; p\right) - (\rho + \delta) \right) c^N \tag{21}$$

$$\dot{c}^S = \left(G_{k^S}\left(k^S; p\right) - (\rho + \delta) \right) c^S \tag{22}$$

$$\dot{k}^N = \left(G_{k^N}\left(k^N; p\right) - \delta \right) k^N + G_{L^N}\left(k^N; p\right) - c^N \tag{23}$$

$$\dot{k}^S = \left(G_{k^S}\left(k^S; p\right) - \delta \right) k^S + G_{L^S}\left(k^S; p\right) - c^S \tag{24}$$

$G_{K^N} \neq G_{K^S}$ and $G_{L^N} \neq G_{L^S}$. The aggregate production function in each country $G\left(k^i; p\right)$ will inherit the properties of the production function(s) active in each sector.

Dynamics

Steady State

Stability: Linearize the system in the neighborhood of the steady state.

From the resulting characteristic equation solve for the 4 eigenvalues.

Specialized steady state

From the laws of motion of the state and co-state variables, we can solve for the steady state of this economy $\{r^{N*}, r^{S*}, w^{N*}, w^{S*}, k^{N*}, k^{S*}, c^{N*}, c^{S*}\}$. The stationary point of the system is:

$$k^{N*} = \left(\frac{\tilde{\gamma}(1-\theta_2)+\gamma(\theta_1-\theta_2)}{(1-\tilde{\gamma})(1-\theta_2)}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \quad (25)$$

$$k^{S*} = \left(\frac{\theta_2}{1-\theta_2}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \quad (26)$$

$$c^{N*} = \left(\rho+\frac{(\rho+\delta)(1-\tilde{\gamma})(1-\theta_2)}{\tilde{\gamma}(1-\theta_2)+\gamma(\theta_1-\theta_2)}\right)k^{N*} \quad (27)$$

$$c^{S*} = \left(\frac{\rho+(1-\theta_2)\delta}{\theta_2}\right)k^{S*} \quad (28)$$

and FPE.

$$\frac{dX(t)}{dt} = \bar{A}X(t)$$

$$\begin{aligned}
\bar{A} &= \begin{pmatrix} 0 & A \\ -I & B \end{pmatrix} \\
A &= \begin{pmatrix} \frac{dr^N}{dk^N}c^N|^* & \frac{dr^N}{dk^S}c^N|^* \\ \frac{dr^S}{dk^N}c^S|^* & \frac{dr^S}{dk^S}c^S|^* \end{pmatrix} \\
B &= \begin{pmatrix} \frac{dr^N}{dk^N}k^N|^* + \rho + \frac{dw^N}{dk^N}|^* & \frac{dr^N}{dk^S}k^N|^* + \frac{dw^N}{dk^S}|^* \\ \frac{dr^S}{dk^N}k^S|^* + \frac{dw^S}{dk^N}|^* & \frac{dr^S}{dk^S}k^S|^* + \rho + \frac{dw^S}{dk^S}|^* \end{pmatrix}
\end{aligned}$$

The system is saddle path stable and has two negative eigenvalues.

Two trajectories leading to the steady state.

Routh-Hurwitz Criteria

Vieta'sTheorem

Let us label the roots in ascending order as ψ_1, ψ_2, ψ_3 and ψ_4 where the first two are the stable ones.

Two paths that can lead us to the steady state.

Pike is the one associated to the less negative root (ψ_2).

Backroad with the most negative root (ψ_1).

Convergence through the pike is slower than through the backroad and the path of adjustment of capital stocks are toward the pike since as $t \longrightarrow \infty$ the eigenvalue that dominates the system is ψ_2.

We can solve for the slope of the pike as a function of the eigenvalue and it is given by the following expression:

$$S(\psi_2) = \frac{(\psi_2)^2 - f\psi_2 + a}{\psi_2 g - b}$$

where a, b, f and g are described in the appendix.

Proposition 7 *The slope of the pike is lower than the slope of the lower bound of the cone of diversification.*

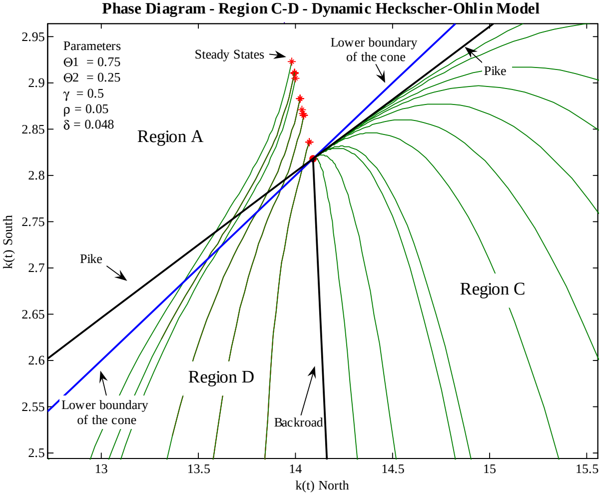

Phase Diagram - Region C-D - Dynamic Heckscher-Ohlin Model
Parameters
Θ1 = 0.75
Θ2 = 0.25
γ = 0.5
ρ = 0.05
δ = 0.048
Steady States
Lower boundary of the cone
Pike
Region A
Pike
Region C
Region D
Lower boundary of the cone
Backroad
k(t) South
k(t) North
2.95
2.9
2.85
2.8
2.75
2.7
2.65
2.6
2.55
2.5
13
13.5
14
14.5
15
15.5

Figure 4

Lemma (4)

Trajectories with initial conditions belonging to the region B below **p** *leave the cone in finite time and converge to the specialized steady state through the Pike from region C.*

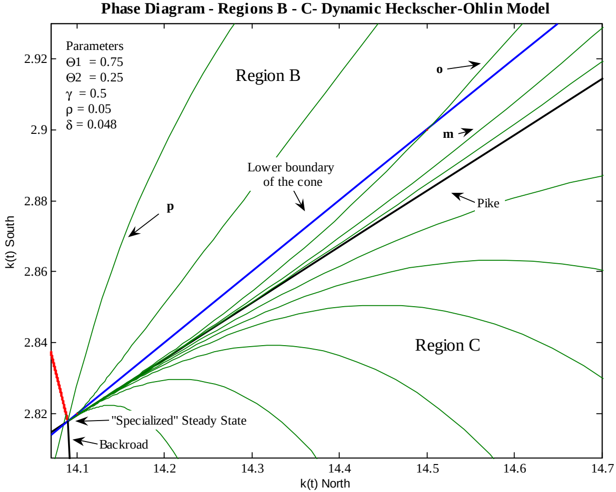

Phase Diagram - Regions B - C- Dynamic Heckscher-Ohlin Model
Parameters
Θ1 = 0.75
Θ2 = 0.25
γ = 0.5
ρ = 0.05
δ = 0.048
Region B
o
m
p
Lower boundary
of the cone
Pike
Region C
"Specialized" Steady State
Backroad
2.92
2.9
2.88
2.86
2.84
2.82
14.1
14.2
14.3
14.4
14.5
14.6
14.7
k(t) South
k(t) North

Figure 5

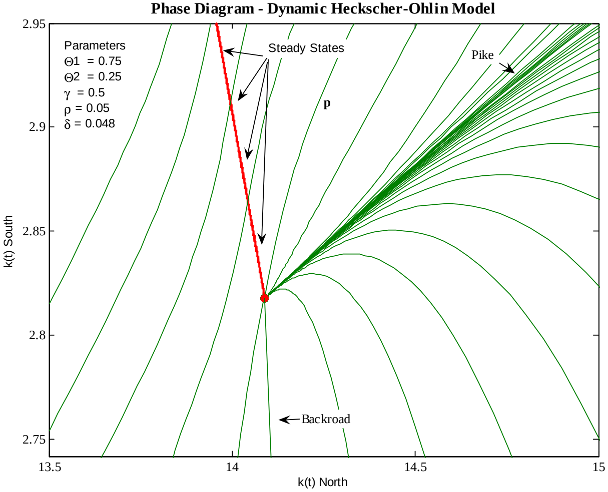
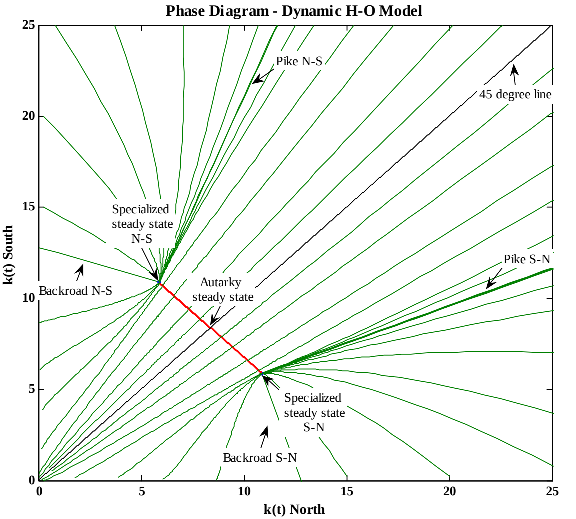

Phase Diagram - Dynamic Heckscher-Ohlin Model
Parameters
Θ1 = 0.75
Θ2 = 0.25
γ = 0.5
ρ = 0.05
δ = 0.048
Steady States
Pike
p
Backroad
k(t) South
k(t) North
2.95
2.9
2.85
2.8
2.75
13.5
14
14.5
15

Figure 6


Phase Diagram - Dynamic H-O Model
Pike N-S
45 degree line
Specialized
steady state
N-S
Backroad N-S
Autarky
steady state
Pike S-N
Specialized
steady state
S-N
Backroad S-N
k(t) South
k(t) North
0
5
10
15
20
25

- Thank You